The integral $H(u,v)$ can be evaluated explicitly, to yield

$$
\begin{aligned}
H(u,v) = uv \Bigg\{ & z\left[\arcsin\frac{uz}{(u^2+v^2)^{1/2}(v^2+z^2)^{1/2}} + \arcsin\frac{vz}{(u^2+v^2)^{1/2}(u^2+z^2)^{1/2}} - \frac{\pi}{2}\right] \\
& - \frac{u}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-v}{(u^2+v^2+z^2)^{1/2}+v} - \frac{v}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-u}{(u^2+v^2+z^2)^{1/2}+u}\Bigg\} \\
& - \frac{u}{2}\left[u(u^2+v^2)^{1/2} + (v^2+z^2)\log\frac{u+(u^2+v^2)^{1/2}}{(v^2+z^2)^{1/2}} - u(u^2+z^2)^{1/2} - z^2\log\frac{u+(u^2+z^2)^{1/2}}{z}\right] \\
& - \frac{v}{2}\left[v(u^2+v^2)^{1/2} + (u^2+z^2)\log\frac{v+(u^2+v^2)^{1/2}}{(u^2+z^2)^{1/2}} - v(v^2+z^2)^{1/2}\right. \\
& \left. - z^2\log\frac{v+(v^2+z^2)^{1/2}}{z}\right] + \frac{1}{3}\left[(u^2+v^2)^{3/2} - (u^2+z^2)^{3/2} - (v^2+z^2)^{3/2} + z^3\right]. \qquad (7)
\end{aligned}
$$

The second kind of integrals appear when one of the partial elements is parallel to the $0xy$ plane, and the second one is parallel to the $0yz$ plane. This is the case when one of the elements belongs to one of the stips, or its image, while the other element belongs to the via, or its image. If the first element vertices are (x_{l1}, y_{l1}, z_1), (x_{u1}, y_{l1}, z_1), (x_{u1}, y_{u1}, z_1), (x_{l1}, y_{u1}, z_1), and the second element vertices are (x_2, y_{l2}, z_{l2}), (x_2, y_{u2}, z_{l2}), (x_2, y_{u2}, z_{u2}), (x_2, y_{l2}, z_{u2}), then the integral to be evaluated has the form

$$
G = \int_{x_{l1}}^{x_{u1}} \int_{z_{l2}}^{z_{u2}} \int_{y_{l1}}^{y_{u1}} \cdot \int_{y_{l2}}^{y_{u2}} \frac{1}{\left[(x-x_2)^2+(y-y')^2+(z_1-z)^2\right]^{1/2}}\, dy'\, dy\, dz\, dx. \qquad (8)
$$

Again, by introducing changes of variables $t = y' - y, t' = y'$, the two integrals over y and y' are reduced to only one integral over t. That integral and the integral over, say x, can be evaluated explicitly, thus yielding

$$
G = \int_{z_1}^{z_2} dz \sum_{p=1}^{2} \sum_{q=1}^{4} a_p b_q H(u_p, v_q, z) \qquad (9)
$$

where

$$
H(u,v,z) = \int_0^u \int_0^v \frac{v-t}{(s^2+t^2+z^2)^{1/2}}\, dt\, ds \qquad (10)
$$

and $a_1 = -1, a_2 = 1, u_1 = |x_{l1} - x_2|, u_2 = |x_{u1} - x_2|$, and b_q and v_q are the same as defined with (6). The integral $H(u,v,z)$ can be integrated explicitly to yield

$$
\begin{aligned}
H(u,v,z) = v \Bigg\{ & z\left[\arcsin\frac{uz}{(u^2+v^2)^{1/2}(v^2+z^2)^{1/2}} \right. \\
& \left. + \arcsin\frac{vz}{(u^2+v^2)^{1/2}(u^2+z^2)^{1/2}} - \frac{\pi}{2}\right] \\
& - \frac{u}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-v}{(u^2+v^2+z^2)^{1/2}+v} \\
& - \frac{v}{2}\log\frac{(u^2+v^2+z^2)^{1/2}-u}{(u^2+v^2+z^2)^{1/2}+u}\Bigg\} \\
& - \frac{1}{2}\left[u(u^2+v^2)^{1/2} + (v^2+z^2)\right. \\
& \cdot\log\frac{u+(u^2+v^2)^{1/2}}{(v^2+z^2)^{1/2}} - u(u^2+z^2)^{1/2} \\
& \left. - z^2\log\frac{u+(u^2+z^2)^{1/2}}{z}\right]. \qquad (11)
\end{aligned}
$$

and the remaining integral in (9) can be evaluated numerically.

Note that due to the reciprocity properties, $L_{ii'} = L_{i'i}$, and thus substantial savings in the computation are possible.

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Analyzing Lossy Radial-Line Stubs

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***Abstract*—Equations for the design and analysis of lossless radial-line stubs are available in the literature. However, when actually fabricated in microstrip or stripline, these stubs possess finite conductor loss. This attenuation must be included if these components are to be properly integrated with other lossy transmission-line elements as part of a micro-**

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wave computer-aided design program, such as Super-Compact®. New equations for open-circuited radial-line stubs, which include the effects of conductor loss, are presented.

I. Introduction

Lossless, radial transmission lines [1], [2] and radial-line stubs [3]–[6] have been discussed in the literature. They have been used for bias lines [3], [4], for impedance matching [5], and in low-pass filters [6]. Equations for the analysis of lossless radial-line stubs have been presented by Vinding [6]. The geometry for the radial-line stub is shown in Fig. 1; r_i is the inner radius, r_o is the stub outer radius, and θ is the angle (in radians) subtended by the stub.

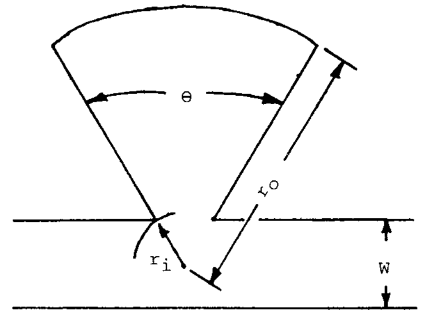

Fig. 1. Radial-line stub connected to a transmission line of width W.

II. New Equations

The equations of Vinding (see Appendix) can be simplified, resulting in the removal of an inverse tangent function and the formulation of a relatively simple equation. This new equation is more appropriate than Vinding's equations for the calculation of the complex input impedance of a lossy radial-line stub. Using the relationships

$$J_m(x) = \sqrt{J_m^2(x) + N_m^2(x)}\cos\left[\tan^{-1}\left(\frac{N_m(x)}{J_m(x)}\right)\right] \quad (1)$$

$$N_m(x) = \sqrt{J_m^2(x) + N_m^2(x)}\sin\left[\tan^{-1}\left(\frac{N_m(x)}{J_m(x)}\right)\right] \quad (2)$$

and the identity

$$\tan^{-1}\left(\frac{-1}{x}\right) + \tan^{-1}(x) = \frac{\pi}{2} \quad (3)$$

the radial-line stub's input impedance can be written as

$$Z_{in} = -j\frac{120\pi h}{r_i\theta\sqrt{\epsilon_r}}\cot(kr_i, kr_0). \quad (4)$$

In (1) and (2), $J_m(x)$ is the Bessel function of the first kind of order m and argument x, and $N_m(x)$ is the Neumann function (Bessel function of the second kind) of order m. In (4), h is the thickness of the microstrip substrate, ϵ_r is the substrate relative permittivity and $\cot(kr_i, kr_o)$ is the large radial cotangent function given by [2]

$$\cot(kr_i, kr_o) = \frac{N_o(kr_i)J_1(kr_o) - J_o(kr_i)N_1(kr_o)}{J_1(kr_i)N_1(kr_o) - N_1(kr_i)J_1(kr_o)}. \quad (5)$$

In (4) and (5), k is the phase constant in radians/unit length.

Defining

$$Z_0(r_i) \equiv \frac{120\pi h}{r_i\theta\sqrt{\epsilon_r}} \quad (6)$$

as the radial-line characteristic impedance at a distance r_i, (4) can be rewritten as

$$Z_{in} = -jZ_0(r_i)\cot(kr_i, kr_o). \quad (7)$$

This equation is of the same form as the familiar equation for the input impedance of a lossless, open-circuited, uniform transmission line, i.e.,

$$Z_{in\,(uniform)} = -jZ_0\cot(kl). \quad (8)$$

Equation (7) is valid only for an ideal, lossless, radial-line stub. Modifications to include the effects of attenuation can be easily accomplished, so long as the losses are not very large. Substitute for k, a complex propagation constant defined by

$$jk = \alpha + j\beta \quad (9a)$$

or

$$k = \beta - j\alpha \quad (9b)$$

where β is the phase constant and α is the attenuation constant for the lossy radial transmission line.

If losses are small

$$\beta = \frac{2\pi\sqrt{\epsilon_r}}{\lambda_0} \quad (10a)$$

and

$$\alpha = \frac{R(r)}{2Z_0(r)} \quad (10b)$$

where λ_0 is the free-space wavelength.

Equation (10b) does not include radiative losses nor the dielectric loss contribution. The latter is usually very small compared to ohmic losses.

Due to the nonuniform nature of the radial line, the series resistance per unit length $R(r)$ is a function of the radial distance. The series resistance per unit length can be expressed as

$$R(r) = \frac{2R_s}{r\theta} \quad (11)$$

where R_s is the conductor surface resistivity in ohms per square.

From (6), the characteristic impedance at r is

$$Z_0(r) = \frac{120\pi h}{r\theta\sqrt{\epsilon_r}}. \quad (12)$$

Combining (10b) through (12) yields

$$\alpha = \frac{R_s\sqrt{\epsilon_r}}{120\pi h}. \quad (13)$$

This result is valid in planar circuits when the conductor loss is considered small or reasonable. It would not be valid for nichrome or tantalum nitride conductor metallizations.

In order to apply this result, the Bessel and Neumann functions must be evaluated for complex arguments instead of real arguments as normally encountered.

The Bessel function of a complex argument can be determined by employing Neumann's addition theorem [7]

$$J_m(u + jv) = \sum_{n=-\infty}^{\infty} J_{m-n}(u)J_n(jv) \quad (14a)$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n J_{m-n}(u)I_n(v). \quad (14b)$$

The identical formulation exists for Neumann functions, i.e.,

$$N_m(u + jv) = \sum_{n=-\infty}^{\infty} (-1)^n N_{m-n}(u)I_n(v) \quad (15)$$

where $I_n(v)$ is the modified Bessel function of the first kind which, by definition [8], is

$$I_n(v) \equiv (-1)^{-n} J_n(jv). \tag{16}$$

For small arguments ($V<1$) [9]

$$I_{-n}(v) = I_n(v) = \frac{1}{n!}\left(\frac{v}{2}\right)^n. \tag{17}$$

Combining (14b), (16), and (17), using only the $n=-1$, $n=0$, and $n=+1$ terms of the summation in (14b), and using the equality

$$J_{-n}(v) = (-1)^n J_n(v) \tag{18}$$

results in

$$J_m(u+jv) \approx J_m(u) + \frac{jv}{2}\left[J_{m-1}(u) - J_{m+1}(u)\right] \tag{19a}$$

$$= J_m(u) + jvJ_m'(u). \tag{19b}$$

Equation (19b) is derived from (19a) by utilizing the recurrence relation [10]

$$J_{m-1}(u) - J_{m+1}(u) = 2J_m'(u). \tag{20}$$

$J_m'(u)$ is the derivative of $J_m(u)$ with respect to u. Similarly

$$N_m(u+jv) = N_m(u) + jvN_m'(u). \tag{21}$$

Using two additional recurrence relationships

$$J_0'(u) = -J_1(u) \tag{22a}$$

and

$$J_1'(u) = J_0(u) - \frac{1}{u}J_1(u) \tag{22b}$$

the resulting equations with the substitutions $u=\beta r$ and $v=-\alpha r$ are

$$J_0(kr) = J_0(\beta r) + j\alpha r J_1(\beta r) \tag{23a}$$

$$J_1(kr) = J_1(\beta r) - j\alpha r\left[J_0(\beta r) - \frac{J_1(\beta r)}{\beta r}\right]. \tag{23b}$$

For the Neumann functions

$$N_0(kr) = N_0(\beta r) + j\alpha r N_1(\beta r) \tag{23c}$$

$$N_1(kr) = N_1(\beta r) - j\alpha r\left[N_0(\beta r) - \frac{N_1(\beta r)}{\beta r}\right]. \tag{23d}$$

The final analysis equations for radial-line stubs with attenuation result from substituting (23a) through (23d), with the appropriate arguments, into (5) and the latter, in turn, into (4).

III. Recommendations for Stripline and Microstrip

For stripline, the dimension h in (4), (6), (12), and (13) should be replaced by the ground-plane spacing b.

For microstrip, it is recommended[1] that the relative dielectric constant ϵ_r should be replaced by an effective dielectric constant ϵ_{eff} calculated [11] for a microstrip of constant width w, where

$$w = (r_i + r_o)\sin\left(\frac{\theta}{2}\right). \tag{24}$$

A computer program has been written to test these equations and to compare the results with the lossless formulation of Vinding. For perfect conductors ($R_s = 0$), the results were identical. For finite values of surface resistance, the equations correctly calculated both the resistive and reactive portions of the input impedance.

[1]This approximation, while providing reasonably accurate results, is not the formulation for effective dielectric constant used in Super-Compact.

IV. Conclusion

New equations, useful for the accurate calculation of the complex input impedance of lossy, radial-line stubs, have been presented. This should lead to an improvement in the accuracy of the predicted performance of circuits which contain these elements.

Appendix

For completeness, the equations due to Vinding, using the notation of this paper, are included below.

$$Z_{in} = j\frac{Z_0(kr_i)h}{r_i}\frac{\cos[\theta(kr_i) - \psi(kr_o)]}{\sin[\psi(kr_i) - \psi(kr_o)]} \tag{A1}$$

$$Z_0(kr_i) = \frac{120\pi}{\sqrt{\epsilon_r}}\left[\frac{J_0^2(kr_i) + N_0^2(kr_i)}{J_1^2(kr_i) + N_1^2(kr_i)}\right]^{1/2} \tag{A2}$$

$$\theta(kr_i) = \tan^{-1}\left[N_0(kr_i)/J_0(kr_i)\right] \tag{A3}$$

$$\psi(kr_i) = \tan^{-1}\left[-J_1(kr_i)/N_1(kr_i)\right] \tag{A4}$$

$$\psi(kr_o) = \tan^{-1}\left[-J_1(kr_o)/N_1(kr_o)\right]. \tag{A5}$$

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Plot of Modal Field Distribution in Rectangular and Circular Waveguides

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The earliest plots of modal field distribution in rectangular/circular waveguides were given by Southworth (1936) [1], Barrow (1936) [2], Schelkunoff (1937) [3], and Chu and Barrow (1937) [4].

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